ON A SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE.

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1. It is the purpose of this note to investigate the problem of integrating the system

$$
\begin{aligned}
&\frac{\partial^2\theta}{\partial u_1\partial u_2}+\frac{\partial\theta}{\partial u_1}-\frac{\partial\theta}{\partial u_2}+3\theta=0,\\
(1)\qquad&\frac{\partial^2\theta}{\partial u_2\partial u_3}+\frac{\partial\theta}{\partial u_2}-\frac{\partial\theta}{\partial u_3}+3\theta=0,\\
&\frac{\partial^2\theta}{\partial u_3\partial u_1}+\frac{\partial\theta}{\partial u_3}-\frac{\partial\theta}{\partial u_1}+3\theta=0,
\end{aligned}
$$

which occurs in the preceding paper.* We begin by showing, by means of the Riemann integration method, that a solution $\theta = \theta\ (u_1, u_2, u_3)$ of (1) is uniquely determined by its values on the three coördinate axes, viz., $\theta\ (u_1, 0, 0)$, $\theta\ (0, u_2, 0)$ and $\theta\ (0, 0, u_3)$.

2. To integrate an equation

$$
(2)\qquad \frac{\partial^2 z}{\partial x\partial y}+a(x, y)\frac{\partial z}{\partial x}+b(x, y)\frac{\partial z}{\partial y}+c(x, y)z=0
$$

by the Riemann method, we form the adjoint equation

$$
(3)\qquad \frac{\partial^2 t}{\partial\xi\partial\eta}-\frac{\partial}{\partial\xi}(a(\xi, \eta)t)-\frac{\partial}{\partial\eta}(b(\xi, \eta)t)+c(\xi, \eta)t=0,
$$

and determine a solution

$$t = t(\xi, \eta;\ x, y)$$

by the initial conditions

$$
(4)\qquad
\begin{aligned}
t &= e^{\int_x^\xi b(\xi, y)d\xi} &&\text{for}\quad \eta = y,\\
t &= e^{\int_y^\eta a(x, \eta)d\eta} &&\text{for}\quad \xi = x;
\end{aligned}
$$

then the solution of (2) which takes the given values $z(x, 0)$ on the x-axis and $z(0, y)$ on the y-axis is uniquely determined, and is given by

$$z(x, y) = t(x, 0;\ x, y)z(x, 0) + t(0, y;\ x, y)z(0, y) - t(0, 0;\ x, y)z(0, 0)$$

* L. P. Eisenhart, Triply conjugate systems with equal point invariants.

$$(5) \qquad + \int_0^x \left[b(\xi, 0) t(\xi, 0;\, x, y) - \frac{\partial}{\partial \xi} t(\xi, 0;\, x, y) \right] z(\xi, 0) d\xi$$

$$+ \int_0^y \left[a(0, \eta) t(0, \eta;\, x, y) - \frac{\partial}{\partial \eta} t(0, \eta;\, x, y) \right] z(0, \eta) d\eta.$$

3. We now apply this to the first of equations (1). The adjoint equation is

$$\frac{\partial^2 t}{\partial v_1 \partial v_2} - \frac{\partial t}{\partial v_1} + \frac{\partial t}{\partial v_2} + 3t = 0$$

and the conditions (4) become

$$t = e^{\int_{u_1}^{v_1} - dv_1} = e^{u_1 - v_1} \qquad \text{for} \qquad v_2 = u_2,$$

$$t = e^{\int_{u_2}^{v_2} dv_2} = e^{v_2 - u_2} \qquad \text{for} \qquad v_1 = u_1.$$

Writing $t = e^{u_1 - u_2} \cdot e^{v_2 - v_1} \varphi$, the adjoint equation becomes

$$\frac{\partial^2 \varphi}{\partial v_1 \partial v_2} + 4\varphi = 0,$$

with the initial conditions $\varphi = 1$ for $v_1 = u_1$ and for $v_2 = u_2$. With the notation

$$(6) \qquad f_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^n}{n!(n+\nu)!} = \frac{1}{(2\sqrt{x})^\nu} J_\nu(4\sqrt{x}),$$

where J_ν is the Bessel function of order ν, it is readily shown that

$$(7) \qquad \varphi(v_1, v_2;\, u_1, u_2) = f_0((u_1 - v_1)(u_2 - v_2))$$

satisfies both the adjoint equation and the initial conditions. From (6) and (7) it follows that

$$\frac{\partial}{\partial v_1} \varphi(v_1, 0;\, u_1, u_2) = 4u_2 f_1(u_2(u_1 - v_1)),$$

$$\frac{\partial}{\partial v_2} \varphi(0, v_2;\, u_1, u_2) = 4u_1 f_1(u_1(u_2 - v_2)),$$

and (5) gives

$$\theta(u_1, u_2, u_3) = e^{-u_2} \theta(u_1, 0, u_3) + e^{u_1} \theta(0, u_2, u_3) - e^{u_1 - u_2} f_0(u_1 u_2) \theta(0, 0, u_3)$$

$$(8) \qquad - e^{-u_2} \int_0^{u_1} e^{u_1 - v_1} \cdot 4u_2 f_1(u_2(u_1 - v_1)) \theta(v_1, 0, u_3) dv_1$$

$$- e^{u_1} \int_0^{u_2} e^{v_2 - u_2} \cdot 4u_1 f_1(u_1(u_2 - v_2)) \theta(0, v_2, u_3) dv_2.$$

4. Since the coefficients in (1) are constant, it follows that $\theta(0, u_2, u_3)$ must satisfy the second, and $\theta(u_1, 0, u_3)$ the third of these equations. The application of the Riemann method gives, in the same way as before,

$$
\begin{aligned}
\theta(0, u_2, u_3) &= e^{-u_3}\theta(0, u_2, 0) + e^{u_2}\theta(0, 0, u_3) - e^{u_2-u_3}f_0(u_2u_3)\theta(0, 0, 0)\\
&\quad - e^{-u_3}\int_0^{u_2} e^{u_2-v_2}\cdot 4u_3f_1(u_3(u_2 - v_2))\theta(0, v_2, 0)dv_2\\
&\quad - e^{u_2}\int_0^{u_3} e^{v_3-u_3}\cdot 4u_2f_1(u_2(u_3 - v_3))\theta(0, 0, v_3)dv_3
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
\theta(u_1, 0, u_3) &= e^{-u_1}\theta(0, 0, u_3) + e^{u_3}\theta(u_1, 0, 0) - e^{u_3-u_1}f_0(u_3u_1)\theta(0, 0, 0)\\
&\quad - e^{-u_1}\int_0^{u_3} e^{u_3-u_3}\cdot 4u_1f_1(u_1(u_3 - v_3))\theta(0, 0, v_3)dv_3\\
&\quad - e^{u_3}\int_0^{u_1} e^{v_1-u_1}\cdot 4u_3f_1(u_3(u_1 - v_1))\theta(v_1, 0, 0)dv_1.
\end{aligned}
\tag{10}
$$

Substituting (9) and (10) in (8), we see that $\theta(u_1, u_2, u_3)$ is uniquely determined when $\theta(u_1, 0, 0)$, $\theta(0, u_2, 0)$ and $\theta(0, 0, u_3)$ are given. We may simplify this solution formally by remarking that, since $\theta(u_1, u_2, u_3)$ is a linear functional of the initial values, it may be written as the sum of four solutions

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3, \tag{11}$$

where θ_0 is any particular solution such that $\theta_0(0, 0, 0) = \theta(0, 0, 0)$, and the initial values of θ_1, θ_2 and θ_3 are given by the following table

	θ_1	θ_2	θ_3
on u_1-axis	$\theta(u_1, 0, 0)-\theta_0(u_1, 0, 0)$	0	0
" u_2- "	0	$\theta(0, u_2, 0)-\theta_0(0, u_2, 0)$	0
" u_3- "	0	0	$\theta(0, 0, u_3)-\theta_0(0, 0, u_3)$.

5. A solution θ_0 is readily found by writing

$$\theta_0 = \theta(0, 0, 0)e^{\lambda_1u_1+\lambda_2u_2+\lambda_3u_3} \tag{12}$$

and substituting in (1), we find the following conditions for the constants $\lambda_1, \lambda_2, \lambda_3$:

$$
\begin{aligned}
\lambda_1\lambda_2 + \lambda_1 - \lambda_2 + 3 &= 0,\\
\lambda_2\lambda_3 + \lambda_2 - \lambda_3 + 3 &= 0,\\
\lambda_3\lambda_1 + \lambda_3 - \lambda_1 + 3 &= 0.
\end{aligned}
$$

The general solution of these equations is

$$\lambda_1 = \frac{1}{t}, \qquad \lambda_2 = \frac{1 + 3t}{t - 1}, \qquad \lambda_3 = \frac{1 - 3t}{t + 1}, \tag{13}$$

where t is arbitrary. For instance, making $t = \pm i\sqrt{3}$ and adding the two resulting expressions (12), we may write

$$\theta_0 = \theta(0, 0, 0) \cos \sqrt{3}(u_1 + u_2 + u_3).$$

6. Writing $\theta(u_1, 0, 0) - \theta_0(u_1, 0, 0) = h(u_1)$ so that $h(0) = 0$, the solution $\theta_1(u_1, u_2, u_3)$ has the values $h(u_1)$, 0, 0 on the three coördinate axes. Equation (9) gives $\theta_1\,(0, u_2, u_3) = 0$, (10) becomes

$$\theta_1(u_1, 0, u_3) = e^{u_3}h(u_1) - e^{u_3}\int_0^{u_1} e^{v_1-u_1}\cdot 4u_3 f_1(u_3(u_1 - v_1)h(v_1)dv_1,$$

and substituting in (8), we find

$$\begin{aligned}\theta_1(u_1, u_2, u_3) = e^{u_3-u_2}\Big\{ & h(u_1) - 4u_3\int_0^{u_1} e^{v_1-u_1} f_1(u_3(u_1 - v_1))h(v_1)dv_1 \\ & - 4u_2\int_0^{u_1} e^{u_1-v_1} f_1(u_2(u_1 - v_1))h(v_1)dv_1 \\ & + 16u_2u_3\int_0^{u_1} e^{u_1-w_1} f_1(u_2(u_1-w_1))dw_1\int_0^{w_1} e^{v_1-w_1} f_1(u_3(w_1-v_1))h(v_1)dv_1\Big\}.\end{aligned}$$

Changing the order of integration in the repeated integral, this becomes

$$(14)\qquad \theta_1(u_1, u_2, u_3) = e^{u_3-u_2}\left[h(u_1) + \int_0^{u_1}\Phi(u_1 - v_1, u_2, u_3)h(v_1)dv_1\right]$$

where

$$\begin{aligned}\Phi(u_1 - v_1, u_2, u_3) = {} & 16u_2u_3\int_{v_1}^{u_1} e^{v_1-w_1} f_1(u_3(w_1 - v_1))f_1(u_2(u_1 - w_1))dw_1 \\ & - 4u_2e^{u_1-v_1}f_1(u_2(u_1 - v_1)) - 4u_3e^{v_1-u_1}f_1(u_3(u_1 - v_1))\end{aligned}$$

or, making $w_1 = v_1 + (u_1 - v_1)x$,

$$(15)\qquad \begin{aligned}\Phi(u_1 - v_1, u_2, u_3) = {} & 16u_2u_3(u_1 - v_1) \\ & \times\int_0^1 e^{-(u_1-v_1)x} f_1(u_3(u_1 - v_1)x)f_1(u_2(u_1 - v_1)(1 - x))dx \\ & - 4u_2e^{u_1-v_1}f_1(u_2(u_1 - v_1)) - 4u_3e^{v_1-u_1}f_1(u_3(u_1 - v_1)).\end{aligned}$$

The corresponding expressions for θ_2 and θ_3 are found by cyclic permutation of the variables.

7. In some cases, $\theta_1(u_1, u_2, u_3)$ may be expressed as a contour integral. From (12) and (13) it follows that a solution of (1) is given by

$$(16)\qquad \theta_1(u_1, u_2, u_3) = \frac{1}{2\pi i}\int_C e^{\frac{u_1}{t} + \frac{1+3t}{t-1}u_2 + \frac{1-3t}{t+1}u_3} f(t)dt,$$

where the contour C encloses $t = 0$, but neither $t = 1$ nor $t = -1$, and $f(t)$

is holomorphic inside C. For $u_1 = 0$, the integrand is holomorphic inside C, so that $\theta_1(0, u_2, u_3) = 0$ and in particular $\theta_1(0, u_2, 0) = \theta_1(0, 0, u_3) = 0$. For $u_2 = u_3 = 0$, we find

$$h(u_1) = \theta_1(u_1, 0, 0) = \frac{1}{2\pi i}\int_C e^{u_1/t} f(t)\,dt.$$

Let $f(t) = \sum_0^\infty c_\nu t^\nu$ converge for $|t| < r$ and deform C into the circle $|t| = \rho < r$; then

$$h(u_1) = \frac{1}{2\pi i}\int_{|t|=\rho} \sum_{\mu,\,\nu=0}^{\infty} \frac{c_\nu u_1{}^\mu}{\mu!} t^{\nu-\mu} dt = \sum_{\nu=0}^{\infty} \frac{c_\nu u_1{}^{\nu+1}}{(\nu+1)!}.$$

When

$$h(u_1) = \sum_{\nu=1}^{\infty} a_\nu u_1{}^\nu$$

is given, it follows that $c_\nu = (\nu + 1)!a_{\nu+1}$ and

$$f(t) = \sum_{\nu=0}^{\infty} (\nu + 1)!a_{\nu+1} t^\nu, \tag{17}$$

provided that this series has a radius of convergence greater than zero. For instance, when $h(u_1) = e^{u_1} - 1$, then $f(t) = 1/(1 - t)$.